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A NOTE ON ONE-DIMENSIONAL ATTRACTING SETS IN THE THREE-SPHERE

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ABSTRACT. This paper is an application of Williams' results for one-dimensional attracting sets to the three-sphere. Our objective is to classify up to Ω-conjugacy all diffeomorphisms of S^3 satisfying Smale's axioms A and B and the condition that the nonwandering set consists of zero- and one-dimensional sinks and sources.

Throughout this paper we will rely repeatedly on the following facts about a sink Λ, $\dim \Lambda = r$, of an Anosov-Smale diffeomorphism f of a manifold M^n. They are a special case of [**3**, §4], and are equivalent to the claims in [**1**, §1.3].

Λ has an open neighborhood N, called *fundamental*, such that

(1) $\bigcap_{k=0}^{\infty} f^k(N) = \Lambda$, $\bigcap_{k=0}^{\infty} f^{-k}(N) = W^s(\Lambda)$,
(2) bd N has a tubular neighborhood V such that $V \cap \Omega(f) = \varnothing$,
(3) $f^{k+1}(N) \subset f^k(N)$ for all k,
(4) N has a foliation $\mathscr{G}N$ by smooth $(n-r)$-cells, G, transverse to Λ,
(5) $K = N/\mathscr{G}N$ is a smooth branched r-manifold, and
(6) if $p: N \to K$ is the quotient, p is a homotopy equivalence.

LEMMA 1. *Let M^n be a compact, connected, oriented manifold and let f be an Anosov-Smale diffeomorphism of M. If $\Omega(f)$ consists of sinks $\Lambda_1, \Lambda_2, \cdots, \Lambda_s$ and sources $\Lambda_{s+1}, \cdots, \Lambda_t$, and* codim $\Lambda_j \geqq 2$, *for all $j = 1, \cdots, t$, then $s=1$ and $t=2$, and Λ_1* and *Λ_2 are connected.*

PROOF. We claim first that if Λ is any sink, it has at most finitely many components. If not, $\mathrm{Per}(f/\Lambda)$ cannot be contained in any finite set of components, since it is dense in Λ. Every component which contains a periodic point is a basic attractor of some iterate of f, and therefore has a neighborhood disjoint from the other components. In this way construct an open cover of M with no finite subcover.

Let $A = \bigcup_{j=1}^{s} \Lambda_j$ and $A^* = \Omega(f) - A$. We claim that $M - A$ is connected.

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By Alexander duality [**4**, p. 296] and the exact homology sequence of a pair

$$\check{H}^{n-1}(A) \cong H_1(M, M - A) \cong \bar{H}_0(M - A),$$

where $\bar{H}_*$ is augmented homology, and $\check{H}^*$ of a closed set is the direct limit of singular cohomology over a cofinal family of open neighborhoods. An easy computation using fundamental neighborhoods and fact (6) shows that the left side is trivial.

Take a sufficiently high iterate of f so that all components of A^* are basic sets. By fact (1), applied to f^{-1}, the unstable neighborhood of each component is open in $M-A$. The same argument applies to A.

LEMMA 2. *Let f be a north pole-south pole map and let Λ be its sink. If* $\dim \Lambda=1$, *$\Lambda, f/\Lambda$ is a generalized solenoid. If N is a fundamental neighborhood and (K, g) is a presentation, $f_*: H_1(N) \to H_1(N)$ is conjugate via p_* to $g_*: H_1(K) \to H_1(K)$. f_* is not nilpotent.*

PROOF. The first claim is a special case of [**1**, Theorem D]. In the course of the proof, Williams proves that

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "p"'] & N \arrow[d, "p"] \\ K \arrow[r, "g"'] & K \end{tikzcd}$$

commutes, from which the second claim follows.

To prove the last claim, we can assume that K is orientable, or else we take the double covering. Then, Williams shows [**2**, Theorem E] that, for some positive integer k, (K, g^k) is shift equivalent to $(\bar{K}, \bar{g})$, where $\bar{K}$ is elementary. Since $\bar{g}$ is an immersion, no iterate of $\bar{g}_*$ has rank zero. Shift equivalence preserves rank, so the claim follows.

We are ready to prove the classification, which is

THEOREM. *If f is a north pole-south pole map, $\Omega(f)$ consists of two connected basic sets, Λ_1 and Λ_2, a sink, and a source, resp., and*

(a) $\dim \Lambda_1=\dim \Lambda_2=\dim \Omega(f)$,

(b) *if* $\dim \Omega(f)=0$, *$\Omega(f)$ consists of two fixed points, and*

(c) *if* $\dim \Omega(f)=1$, *$\Lambda_1, f/\Lambda_1$ and $\Lambda_2, f^{-1}/\Lambda_2$ are generalized solenoids. If (K_1, g_1) and (K_2, g_2) are presentations, resp., $H_1(K_1)$ and $H_1(K_2)$ are isomorphic free groups, and under the isomorphism $g_{1*} = (g_{2*})^t$.*

PROOF. The first assertion follows from Lemma 1. Let N be a fundamental neighborhood of Λ_1; $M=\operatorname{int}(S^3-N)$ is a fundamental neighborhood of Λ_2. Let $N_k=f^k(N)$ and $M_k=f^k(M)$. By Alexander duality and the exact homology sequence of a pair, for all integers k,

(i) $\check{H}^1(\mathrm{cl}\, N_k) \simeq H_2(S^3, M_k) \simeq H_1(M_k)$.

Since bd N has a tubular neighborhood in S^3, by [**4**, p. 290] and standard arguments,

(ii) $\check{H}^1(\mathrm{cl}\, N_k) \simeq H^1(\mathrm{cl}\, N_k) \simeq H^1(N_k)$.

$H^1(N_k)$ is free finitely generated, by fact (6), so by the universal coefficient theorem for cohomology [**4**, p. 248],

(iii) $H_1(N_k) \simeq \mathrm{Hom}(H^1(N_k); Z)$.

All these isomorphisms are natural. Together they imply that

$$\begin{array}{ccccc} H_1(N_k) & \simeq & H^1(N_k) & \simeq & H_1(M_k) \\ \uparrow i^{*t} & & \updownarrow i^* & & \downarrow i_* \\ H_1(N_{k+1}) & \simeq & H^1(N_{k+1}) & \simeq & H_1(M_{k+1}) \end{array}$$

commutes, where i is the inclusion. By naturality $i^{*t} = i_*$. Claims (a) and (b) follows from observing that, if $\dim \Lambda_1 = 0$, N_k is contractible, so $(f^{-1}/M)_*$ is necessarily nilpotent. Then, from Lemma 2 we know $\dim \Lambda_2 = 0$.

The last claim requires a computation. We will show that if $(f/N_k)_* : H_1(N_k) \to H_1(N_k)$ and $(f^{-1}/M_{k+1}) : H_1(M_{k+1}) \to H_1(M_{k+1})$, $(f/N_k)_* = (f^{-1}/M_{k+1})^t_*$. Choose bases for the groups in the above diagram for which the matrix representations of these maps are equal, respectively, to those of $i_* : H_1(N_{k+1}) \to H_1(N_k)$ and $i_* : H_1(M_k) \to H_1(M_{k+1})$. The claim follows from the diagram. These matrices are invariant of k, so we are done.

Bibliography

1. R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–487. MR **36** #897.

2. ———, *Classification of one-dimensional attractors*, Northwestern University, Evanston, Ill. (mimeographed notes).

3. M. Hirsch et al., *Neighborhoods of hyperbolic sets*, Invent. Math. **9** (1969/70), 121–134. MR **41** #7232.

4. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR **35** #1007.

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