

A NOTE ON ONE-DIMENSIONAL ATTRACTING SETS  
IN THE THREE-SPHERE

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ABSTRACT. This paper is an application of Williams' results for one-dimensional attracting sets to the three-sphere. Our objective is to classify up to  $\Omega$ -conjugacy all diffeomorphisms of  $S^3$  satisfying Smale's axioms A and B and the condition that the nonwandering set consists of zero- and one-dimensional sinks and sources.

Throughout this paper we will rely repeatedly on the following facts about a sink  $\Lambda$ ,  $\dim \Lambda = r$ , of an Anosov-Smale diffeomorphism  $f$  of a manifold  $M^n$ . They are a special case of [3, §4], and are equivalent to the claims in [1, §1.3].

$\Lambda$  has an open neighborhood  $N$ , called *fundamental*, such that

- (1)  $\bigcap_{k=0}^{\infty} f^k(N) = \Lambda$ ,  $\bigcap_{k=0}^{\infty} f^{-k}(N) = W^s(\Lambda)$ ,
- (2)  $\text{bd } N$  has a tubular neighborhood  $V$  such that  $V \cap \Omega(f) = \emptyset$ ,
- (3)  $f^{k+1}(N) \subset f^k(N)$  for all  $k$ ,
- (4)  $N$  has a foliation  $\mathcal{G}N$  by smooth  $(n-r)$ -cells,  $G$ , transverse to  $\Lambda$ ,
- (5)  $K = N/\mathcal{G}N$  is a smooth branched  $r$ -manifold, and
- (6) if  $p: N \rightarrow K$  is the quotient,  $p$  is a homotopy equivalence.

LEMMA 1. *Let  $M^n$  be a compact, connected, oriented manifold and let  $f$  be an Anosov-Smale diffeomorphism of  $M$ . If  $\Omega(f)$  consists of sinks  $\Lambda_1, \Lambda_2, \dots, \Lambda_s$  and sources  $\Lambda_{s+1}, \dots, \Lambda_t$ , and  $\text{codim } \Lambda_j \geq 2$ , for all  $j = 1, \dots, t$ , then  $s = 1$  and  $t = 2$ , and  $\Lambda_1$  and  $\Lambda_2$  are connected.*

PROOF. We claim first that if  $\Lambda$  is any sink, it has at most finitely many components. If not,  $\text{Per}(f/\Lambda)$  cannot be contained in any finite set of components, since it is dense in  $\Lambda$ . Every component which contains a periodic point is a basic attractor of some iterate of  $f$ , and therefore has a neighborhood disjoint from the other components. In this way construct an open cover of  $M$  with no finite subcover.

Let  $A = \bigcup_{j=1}^s \Lambda_j$  and  $A^* = \Omega(f) - A$ . We claim that  $M - A$  is connected.

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By Alexander duality [4, p. 296] and the exact homology sequence of a pair

$$\check{H}^{n-1}(A) \cong H_1(M, M - A) \cong \bar{H}_0(M - A),$$

where  $\bar{H}_*$  is augmented homology, and  $\check{H}^*$  of a closed set is the direct limit of singular cohomology over a cofinal family of open neighborhoods. An easy computation using fundamental neighborhoods and fact (6) shows that the left side is trivial.

Take a sufficiently high iterate of  $f$  so that all components of  $A^*$  are basic sets. By fact (1), applied to  $f^{-1}$ , the unstable neighborhood of each component is open in  $M - A$ . The same argument applies to  $A$ .

LEMMA 2. *Let  $f$  be a north pole-south pole map and let  $\Lambda$  be its sink. If  $\dim \Lambda = 1$ ,  $\Lambda, f|_{\Lambda}$  is a generalized solenoid. If  $N$  is a fundamental neighborhood and  $(K, g)$  is a presentation,  $f_* : H_1(N) \rightarrow H_1(N)$  is conjugate via  $p_*$  to  $g_* : H_1(K) \rightarrow H_1(K)$ .  $f_*$  is not nilpotent.*

PROOF. The first claim is a special case of [1, Theorem D]. In the course of the proof, Williams proves that

$$\begin{array}{ccc} N & \xrightarrow{f} & N \\ p \downarrow & & \downarrow p \\ K & \xrightarrow{g} & K \end{array}$$

commutes, from which the second claim follows.

To prove the last claim, we can assume that  $K$  is orientable, or else we take the double covering. Then, Williams shows [2, Theorem E] that, for some positive integer  $k$ ,  $(K, g^k)$  is shift equivalent to  $(\bar{K}, \bar{g})$ , where  $\bar{K}$  is elementary. Since  $\bar{g}$  is an immersion, no iterate of  $\bar{g}_*$  has rank zero. Shift equivalence preserves rank, so the claim follows.

We are ready to prove the classification, which is

THEOREM. *If  $f$  is a north pole-south pole map,  $\Omega(f)$  consists of two connected basic sets,  $\Lambda_1$  and  $\Lambda_2$ , a sink, and a source, resp., and*

- (a)  $\dim \Lambda_1 = \dim \Lambda_2 = \dim \Omega(f)$ ,
  - (b) if  $\dim \Omega(f) = 0$ ,  $\Omega(f)$  consists of two fixed points, and
  - (c) if  $\dim \Omega(f) = 1$ ,  $\Lambda_1, f|_{\Lambda_1}$  and  $\Lambda_2, f^{-1}|_{\Lambda_2}$  are generalized solenoids.
- If  $(K_1, g_1)$  and  $(K_2, g_2)$  are presentations, resp.,  $H_1(K_1)$  and  $H_1(K_2)$  are isomorphic free groups, and under the isomorphism  $g_{1*} = (g_{2*})^t$ .*

PROOF. The first assertion follows from Lemma 1. Let  $N$  be a fundamental neighborhood of  $\Lambda_1$ ;  $M = \text{int}(S^3 - N)$  is a fundamental neighborhood of  $\Lambda_2$ . Let  $N_k = f^k(N)$  and  $M_k = f^k(M)$ . By Alexander duality and

the exact homology sequence of a pair, for all integers  $k$ ,

$$(i) \check{H}^1(\text{cl } N_k) \cong H_2(S^3, M_k) \cong H_1(M_k).$$

Since  $\text{bd } N$  has a tubular neighborhood in  $S^3$ , by [4, p. 290] and standard arguments,

$$(ii) \check{H}^1(\text{cl } N_k) \cong H^1(\text{cl } N_k) \cong H^1(N_k).$$

$H^1(N_k)$  is free finitely generated, by fact (6), so by the universal coefficient theorem for cohomology [4, p. 248],

$$(iii) H_1(N_k) \cong \text{Hom}(H^1(N_k); Z).$$

All these isomorphisms are natural. Together they imply that

$$\begin{array}{ccccc} H_1(N_k) & \cong & H^1(N_k) & \cong & H_1(M_k) \\ \uparrow i^{*\iota} & & \downarrow i^* & & \downarrow i_* \\ H_1(N_{k+1}) & \cong & H^1(N_{k+1}) & \cong & H_1(M_{k+1}) \end{array}$$

commutes, where  $i$  is the inclusion. By naturality  $i^{*\iota} = i_*$ . Claims (a) and (b) follows from observing that, if  $\dim \Lambda_1 = 0$ ,  $N_k$  is contractible, so  $(f^{-1}/M)_*$  is necessarily nilpotent. Then, from Lemma 2 we know  $\dim \Lambda_2 = 0$ .

The last claim requires a computation. We will show that if  $(f/N_k)_* : H_1(N_k) \rightarrow H_1(N_k)$  and  $(f^{-1}/M_{k+1})_* : H_1(M_{k+1}) \rightarrow H_1(M_{k+1})$ ,  $(f/N_k)_* = (f^{-1}/M_{k+1})_*$ . Choose bases for the groups in the above diagram for which the matrix representations of these maps are equal, respectively, to those of  $i_* : H_1(N_{k+1}) \rightarrow H_1(N_k)$  and  $i_* : H_1(M_k) \rightarrow H_1(M_{k+1})$ . The claim follows from the diagram. These matrices are invariant of  $k$ , so we are done.

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