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Note

Finite Partitions of Spheres

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Let S^n be the unit sphere in E^{n+1}; an $(n-1)$-sphere in S^n is an intersection of S^n with a hyperplane. A theorem is proved on colorings of sphere which implies the Proposition: If B is an m-coloring of S^2, for $m \geqslant 6$, B4-colors some 1–sphere. If B is an m-coloring of S^3, $m \geqslant 8$; B 5-colors some 2-sphere.

Recent results in geometry on spheres [1] have shed some light on the following combinatorial problem. In the terminology of [2], an m-coloring of the unit n-sphere, S^n, is any partition into m well-defined sets (m a natural number); a coloring is an m-coloring for some m. A subset A of S^n is i-colored if the coloring of A induced by restriction is an i-coloring. Given a coloring B we define the *incidence* of B to be the greatest i for which some (round) $(n-1)$-sphere is i-colored. The problem is to find, for given n and i, the largest m for which every m-coloring has incidence no greater than i. Equivalently, given m and n, find the smallest i such that every m-coloring has incidence at least i.

This problem is related to the class of Euclidean Ramsey problems introduced in [2]. The general Euclidean Ramsey problem on E^n is to characterize triples (H, A, m), for H a subgroup of the group of similarities of E^n, A a finite subset of E^n, and m a natural number, having the property that the orbit of A under H is rich enough to contain a monochromatic element for any m-coloring of E^n. Such a triple is said to have the Ramsey property. To define the equivalent problem on the n-sphere it is only necessary to select a counterpart to the group of similarities. This group is called the Mobius group of S^n and consists of all circle-preserving bijections. There is a natural correspondence between the Mobius group of S^n and the similarities of E^n based on the observation that any similarity can be thought of as a Mobius transformation which leaves $\{\infty\}$ fixed.

The Mobius transformations of S^2 are the homographies, or linear fractional transformations, and the antihomographies, in the terminology of [3].

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Equivalently, they are the conformal bijections of S^2. It is easily shown that a bijection of S^n is circle-preserving if and only if it preserves codimension one spheres, so the two natural generalizations to S^n coincide. The correspondence between the Mobius group and the similarity group in any dimension follows directly from a sterographic projection of E^n into S^n.

The general problem, of which the results of this paper constitute a very special case, can be posed in terms analogous to the Ramsey problem. Given an m-coloring B of S^n, a subgroup H of the Mobius groups, and a set S, each element of the orbit of S under H is colored. Define the *incidence* of B to be the maximum, over the orbit of S, of the color number of these colorings. The problem is to compute, for given H, S, and m, the minimal incidence over all possible m-colorings. Specifically, presented in this paper are some results for the case where H is the full Mobius group and S is the equator.

Let $m = m(i, n)$ denote the maximal cardinality of a coloring of S^n with incidence at most i. This function of i has the following properties, which together imply that the only interesting cases occur where $i = n + 1$:

PROPOSITION A.

(a) *$m(i, n) = i$, if $i \leqslant n$, and*

(b) *$m(i, n) = \infty$, if $i \geqslant n + 2$.*

Proof. (a) $m(i, n) \geqslant i$, trivially. For any set of $n + 1$ or fewer points, there is an $(n - 1)$-sphere through them. If $m > i$ is an integer, the coloring which consists of i singletons $\{p_1\},\ldots, \{p_i\}$ and any $(m - 1)$-coloring of S^n-$\{p_1,\ldots, p_i\}$ has incidence at least $i + 1$. Thus, $m \leqslant i$. Claim (a) follows.

(b) For any m, the m-coloring which consists of $(m - 1)$-singletons $\{p_1\},\ldots, \{p_{m-1}\}$, chosen so that no $n + 2$ of them lie on an $(n - 1)$-sphere, and the set $S^n - \{p_1,\ldots, p_{m-1}\}$ has incidence no greater than $n + 2$. Thus, if $i \geqslant n + 2$, there is no maximal m. Q.E.D.

PROPOSITION B.

(a) $m(2, 1) = \infty$

(b) $4 \leqslant m(3, 2) \leqslant 5$, and

(c) $5 \leqslant m(4, 3) \leqslant 7$.

Proof. (a) Since a 0-sphere on S^1 is an arbitrary pair of points, every coloring has incidence at most two.

(b) The lower bound is achieved by the following 4-coloring: 3-color the equator, and take the fourth set to be the complement of the equator.

The upper bound is a corollary of [1, Theorem 1] which implies the following characterization of Mobius transformations of S^2. If $f: S^2 \to S^2$ is any function mapping circles into circles and if the image of f contains six points, no five of which lie on a circle, f is a Mobius transformation. Suppose now that the 2-sphere admits an m-coloring with incidence three, for $m \geqslant 6$. We assume without loss of generality that $m = 6$. Let $B = \{B_1, B_2, \ldots, B_6\}$ be the coloring, and let $X = \{b_1, b_2, b_3, b_4, b_5, b_6\} c S^2$ be chosen so that no five points lie on a circle. Since X satisfies the image condition of [1, Theorem 1], any circle-preserving map of S^2 with image containing X is a Mobius transformation, and therefore a bijection. Define $f: S^2 \to X$ by $f(B_j) = b_j$, $j = 1, \ldots, 6$. f is circle-preserving trivially, the image of a circle is at most three points. But a contradiction ensues, which proves that B is impossible.

(c) A proof of the lower bound is straightforward. The upper bound is established by proving the impossibility of an 8-coloring with incidence four. The essential step is analogous to part (b) above, and follows from a characterization of Mobius transformations of S^3. From [1, Theorem 2], a function f: $S^3 \to S^3$ which maps circles into circles, and which maps onto an eight point set, no six of which lie on a 2-sphere, is a Mobius transformation. Since any 2-sphere preserving function of S^3 satisfying the image condition is necessarily circle-preserving, as well, the theorem readily translates to 2-sphere, preserving functions. If $B = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8\}$ were an 8-coloring of the 3-sphere having incidence four, there would be a 2-sphere preserving function of S^3 to an eight point set in violation of the theorem.

COROLLARY. *If a 2-sphere is partitioned into six or more sets, some circle meets at least four of them.*

The corollary could be strengthened somewhat. Let B be a 6-coloring of S^2, and suppose no circle is 5-colored. Then there is a triple in B, say (B_1, B_2, B_3), contained in distinct 4-tuples, say (B_1, B_2, B_3, B_4) and (B_1, B_2, B_3, B_5), and circles C and C', so that C is colored by one 4-tuple and C' by the other. The reason is that otherwise we can define a circle-preserving map of S^2 to $X = \{0, 1, i, -1, -i, \infty,\}$ thought of as embedded in S^2 by sterographic projection.

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