(n+3)-Coloring the n-Sphere

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Abstract

We address a combinatorial proposition for the n-sphere and a corresponding proposition in inversive geometry on the n-sphere, and demonstrate the intimate connection between them. Specifically, in terms of combinatorial geometry, we show that any coloring of the n-sphere by n+3 colors must (n+2)-color some (n-1)-sphere. In regard to inversive geometry, we characterize the structure of the class of smallest subsets of the n-sphere that has the property that if T is a well-defined function of the n-sphere that preserves (n–1)-spheres and if the image of T contains a member of this class, T must be an inversive transformation. Lastly, we demonstrate that the combinatorial theorem is equivalent to the theorem that defines this class of sets.

Keywords: circle-preserving map, sphere-preserving map, inversive transformation, coloring the n-sphere, coloring n-space

MSC: 51B10

Open Problem

For natural numbers k and m, where k ≥ 4 and m ≥ 5, consider the m-coloring of the 2-sphere which has the property that no circle on the sphere meets more than k-many colors. We count "circles" in a logical way, where if given set of colors actually occurs on some circle, we say that the corresponding set of integers – the integer names of the colors on that circle – is a "circle" that is colored by that set. Thus for example, if some coloring 3-colors a circle with colors k_1, k_2, and k_3, we say that the triple (k_1, k_2, k_3) is a 3-colored circle. Any count of circles, in this sense, refers to a count not of the underlying circles, but of the corresponding sets of colors. In these terms, the combinatorial proposition stated above is equivalent to "Prop: every 5-coloring of the 2-sphere must have a colored 4-circle."

Proof of this corollary. Let the colors be 1, 2, 3, 4, and 5. There is a 4-colored circle in the sphere. Suppose it contains colors 1, 2, 3, and 4. That 4-tuple must occur as a circle, but no other 4-tuple is required in addition to that one. (4-color the equator and apply the remaining color to both open hemispheres.)

Now for given k and m, let p equal the minimal number of k-circles. p(k,m) is a function and p(4,5) = 1.

Problem: Compute p for arbitrary k and m. It yields the following **Theorem: If k is the largest number of colors on a circle, there must be at least p k-colored circles**

Note that by the way we have defined the term "colored circles," they must correspond to p different combinations of colors. We know a lot about p, but still do not have a formula.